And first indeed, that BK cannot be cut by AL in any point M follows absolutely from Euclid I. 17, since otherwise in the same triangle MKL we would have two right angles at the points K and L, apart from the fact that in this case we would have our assertion about that angle BAN, that it is not in such circumstances the least of all.

But again AL cannot be the continuation of AN; because otherwise in the quadrilateral $NDKL$ we would have four right angles, against the hypothesis of acute angle.

But neither can it cut DN produced in any exterior point H; because angle AHN (from Euclid I. 16) would be acute, on account of the external angle AND supposed right; and therefore angle DHL would be obtuse, and so in the quadrilateral $DHLK$ we would have four angles, which taken together would be greater than four right angles, against the aforesaid hypothesis of acute angle.

Therefore it follows that the angle BAN must be cut by this AL, and therefore cannot be declared the least of all, drawn under which AN has with BX in two distinct points a common perpendicular ND.

Quod erat secundo loco demonstrandum. Itaque constat etc.

Corollary. But hence is permitted to observe, that under a lesser angle BAL is obtained (in hypothesis of acute angle) a common perpendicular LK, more remote indeed from the base AB, as follows from the construction, but moreover less than the other nearer common perpendicular ND, which is obtained under a greater angle BAN.

The reason of this latter is because in the quadrilateral $LKDS$ the angle at the point S is acute in the aforesaid hypothesis, since the three remaining angles are supposed right.

Wherefore (from Corollary I. to Proposition III.) the side LK will be less than the opposite side SD, and so much less than the side ND.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

III.

Construction of a One Parameter Group from an Infinitesimal Transformation.

9. Let there be given the one parameter continuous group

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a); \tag{1}$$

assume further that it contains the inverse transformation of every transformation in it, *i. e.* that the solutions of the equations (1) with regard to x and y have the form

$$x=\varphi(x_1, y_1, b), \quad y=\psi(x_1, y_1, b),$$

in which b is a canstant depending only on a. In the preceding paragraphs the theorem of LIE that every one parameter group whose transformations are inverse in pairs contains an infinitesimal transformation was arrived at both geometrically and analytically. Either process may be formulated symbolically as follows. If T_a represent the transformation of the group corresponding to the parameter a, its inverse T_a^{-1} is also contained in (1) by hypothesis. Further $T_{a+\delta a}$ will represent the transformation corresponding to the parameter $a+\delta a$, and therefore the transformation of the group (1) that differs from T_a by an infinitesimal. The successive application or the product of $T_{a+\delta a}$ and T_a^{-1}, namely $T_{a+\delta a}T_a^{-1}$ (which belongs to the group by virtue of our first supposition that the product of any two transformations of the group is itself a transformation of the group), differs infinitesimally from $T_aT_a^{-1}$, the identical transformation, and hence is itself an infinitesimal transformation belonging to the group (1).

10. On the other hand there is always a completely determinate continuous group of transformations which contains a given infinitesimal transformation. The truth of this assertion may be made to appear symbolically in the following manner.

Let S be any arbitrary transformation in the xy-plane. Construct the transformations which are equivalent to the repetition of S once, twice, and so on to n-times ; also the inverse of S, S^{-1}, and those equivalent to the repetition of this inverse once, twice, and so on to n-times ; we then have an infinite family of transformations,

$$\ldots, S^{-n}, \ldots, S^{-2}, S^{-1}, S^0, S^1, S^2, \ldots, S^n, \ldots,$$

where S^0 is the identical transformation, while n represents every possible positive whole number. This infinite family is a group, since if p and q are two positive or negative numbers, the product of S^p and S^q is equivalent to S^{p+q}, but the group is a discontinuous one.

In this manner, beginning with an arbitrary transformation S an infinite number of discontinuous groups in x and y may be constructed. Passing now to the limiting case, if, in particular, S is an infinitesimal transformation, then S^n and S^{n+1} differ from each other by an infinitesimal, and we have accordingly a continuous group constructed from, and containing the infiniteslmal transformation, S.

11. LIE has invented an ingenious kinematical illustration of this limiting case, which serves as a concrete introduction to the rigorous demonstration of the theorem.

The infinitesimal transformation is defined by two equations of the form

$$x'=x+\xi(x, y)\delta t+\ldots, \quad y'=y+\eta(x, y)\delta t\ldots, \tag{2}$$

where ξ and η are any two given functions of x and y, the quantity δt an infinitesimal, and the terms omitted convergent power series in δt beginning with δt^2.

The coördinates of the transformed point (x', y') differ from those of the original point (x, y) by the infinitesimal increments

$$\delta x = \xi(x, y)\delta t, \quad \delta y = \eta(x, y)\delta t,$$

when terms of the second order of infinitesimals are neglected. The infinitesimal transformation makes correspond to every point (x, y) an infinitesimal arrow (say) whose length is

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{\xi^2 + \eta^2}\, \delta t,$$

and direction
$$\frac{\delta y}{\delta x} = \frac{\eta}{\xi};$$

and in general to different points arrows of different lengths and different directions. The infinitesimal transformation thus puts all the points (x, y) of the plane in motion, and if the variable t be taken as the time, these points describe in the element of time δt, the infinitesimal paths $\sqrt{\xi^2 + \eta^2}\, \delta t$, whose projections on the axes are $\xi\delta t$ and $\eta\delta t$. In the first element of time δt the point (x, y) goes over into (x', y') describing the path $\sqrt{\xi(x, y)^2 + \eta(x, y)^2}\, \delta t$, in the next element δt it runs over the infinitesimal path $\sqrt{\xi(x', y')^2 + \eta(x', y')^2}\, \delta t$, and so on. The original point (x, y) assumes, by the continued application of the infinitesimal transformation, a continuous series of positions which may be represented by a curve. This motion of the points of the plane is characterized by the fact that the components of the velocity of every point (x, y) have the values

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1),$$

which depend only on the position and not on the time. Since the change of position is to repeat itself from moment to moment, the motion is a so-called stationary motion and can be compared to the flow of the particles of a compressible fluid. That the phenomena of a stationary motion exhibit the group property is readily seen, for if the stationary motion carries the points (x, y) to the position (x_1, y_1) in the time t_1 and then these new points (x_1, y_1) to the positions (x_2, y_2) in the time t_2, it is clear that the motion carries the original points (x, y) to the positions (x_2, y_2) in the time $t_1 + t_2$; *i. e.* the successive performance of two transformations (t_1) and (t_2) of the family is equivalent to a single transformation $(t_1 + t_2)$ of the family.

12. This kinematical illustration may now be replaced by the following rigorous analytical reasoning.

The two differential equations

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1), \tag{3}$$

determine x_1 and y_1 as functions of t, and the initial values corresponding to $t=0$

which we take as $x_1=x$, $y_1=y$. In order to determine these functions x_1 and y_1, it is necessary to integrate the simultaneous system

$$\frac{dx_1}{\xi(x_1,y_1)}=\frac{dy_1}{\eta(x_1,y_1)}=dt, \tag{4}$$

with the initial conditions that $x_1=x$ and $y_1=y$ for $t=0$.

This integration is effected as follows· The differential equation in x_1, y_1

$$\frac{dx_1}{\xi(x_1,\, y_1)}=\frac{dy_1}{\eta(x_1,\, y_1)}$$

has an integral, $\Omega(x_1,\, y_1)$, which, since it is free from t; is also an integral of the whole simultaneous system (4). In order to find the second integral of the system which contains t, we eliminate say y_1 between the two equations

$$\Omega(x_1,\, y_1)=\text{constant}=c, \quad \text{and} \quad \frac{dx_1}{\xi(x_1,\, y_1)}=dt,$$

and obtain a differential equation,

$$\frac{dx_1}{\theta(x_1,\, c)}=dt.$$

Since the left hand member of this equation does not contain t it can be integrated by a quadrature* and its integral has the form $f(x_1,\, c)-t$. But this is not an integral of the system (4) until c has been eliminated by means of the equation $\Omega(x_1,\, y_1)=c$. Eliminating c, the second integral of the system (4) appears in the form $W(x_1,\, y_1)-t$.†

Finally, determining the constants of integration by the initial conditions that $x_1=x$, $y_1=y$ for $t=0$, we have as the result of the integration

$$\begin{aligned} \Omega(x_1,\, y_1)&=\Omega(x,\, y), \\ W(x_1,\, y_1)-t&=W(x,\, y). \end{aligned} \tag{5}$$

Without solving these equations for x_1, y_1 it is easy to see that they define a one parameter group, for the transformation of the family (5) which corresponds to the parameter value t carries the points $(x,\, y)$ into the points $(x_1,\, y_1)$, whose coördinates can be found by solving the equations (5) for x_1, y_1. A sec-

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*By the term quadrature is meant an integral of the form $\int F(x)dx$. It is assumed that a quadrature can always be performed.

†The reader will observe that this same integral would have been found had we begun by eliminating x_1 from $\dfrac{dy_1}{\eta(x_1,\, y_1)}=dt$ by means of $\Omega(x_1,\, y_1)=c$. This elimination would have given the differential equation $\dfrac{dy_1}{\lambda(y_1,\, c)}=dt$; the integral of the latter, $\mu(y_1,\, c)-t$, is found by a quadrature; eliminating c by means of $\Omega(x_1,\, y_1)=c$, we have finally the second integral of the system, $W(x_1,\, y_1)-t$.

ond transformation of the same family with the parameter value t_1 will change the points (x_1, y_1) into the points (x_2, y_2) whose coördinates are found from the equations,

$$\begin{aligned} \Omega(x_2, y_2) &= \Omega(x_1, y_1), \\ W(x_2, y_2) - t &= W(x_1, y_1). \end{aligned} \tag{6}$$

In order to find the transformation which carries the original points (x_1, y_1) directly into the final positions (x_2, y_2), it is only necessary to eliminate x_1, y_1 from the equations (5) and (6). The elimination gives at once

$$\Omega(x_2, y_2) = \Omega(x, y),$$
$$W(x_2, y_2) - (t + t_1) = W(x, y).$$

But these equations represent the transformation of the family (5) corresponding to the parameter value $t + t_1$; hence the family (5) possesses the group property. The group contains also the inverse transformation of every transformation in it and the identical transformation.

The equations (5) can be solved with regard to x_1, y_1 in the form

$$x_1 = \Phi(x, y, t), \quad y_1 = \Psi(x, y, t). \tag{7}$$

These two functions can be expanded in powers of t by Maclaurin's theorem. In order to effect the expansion we must have the values

$$\left(\frac{dx_1}{dt}\right)_{t=0}, \quad \left(\frac{d^2 x_1}{dt^2}\right)_{t=0}, \ldots .$$

From equations (4) we have $\frac{dx_1}{dt} = \xi(x_1, y_1)$, with $x_1 = x$, $y_1 = y$, for $t = 0$;

hence,
$$\left(\frac{dx_1}{dt}\right)_{t=0} = \xi(x, y).$$

The equations (4) give also

$$\frac{d^2 x_1}{dt^2} = \frac{\partial \xi(x_1, y_1)}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial \xi(x_1, y_1)}{\partial y_1} \frac{dy_1}{dt}$$
$$= \frac{\partial \xi(x_1, y_1)}{\partial x_1} \xi(x_1, y_1) + \frac{\partial \xi(x_1, y_1)}{\partial y_1} \eta(x_1, y_1) ;$$

hence
$$\left(\frac{d^2 x_1}{dt^2}\right)_{t=0} = \frac{\partial \xi(x, y)}{\partial x} \xi(x, y) + \frac{\partial \xi(x, y)}{\partial y} \eta(x, y).$$

Similarly, $\left(\frac{dy_1}{dt}\right)_{t=0} = \eta(x, y)$, $\left(\frac{d^2 y_1}{dt^2}\right)_{t=0} = \frac{\partial \eta(x, y)}{dx} \xi(x, y) + \frac{\partial \eta(x, y)}{dy} \eta(x, y)$.

Accordingly equations (7) become by Maclaurin's theorem,

$$x_1 = x + \xi(x, y)t + \left(\xi \frac{\partial \xi}{\partial x} + \eta \frac{\partial \xi}{\partial y} \right) \frac{t^2}{1.2} + \ldots,$$
$$y_2 = y + \eta(x, y)t + \left(\xi \frac{\partial \eta}{\partial x} + \eta \frac{\partial \eta}{\partial y} \right) \frac{t^2}{1.2} + \ldots \tag{8}$$

The reader will observe that $t=0$ in the equations (8) gives the identical transformation, and $t=\delta t$ gives an infinitesimal transformation which to terms of the second order agrees with the original infinitesimal transformation (2).

All these facts may now be summed up in the following theorem of LIE: *Every infinitesimal transformation*

$$x_1 = x + \xi(x, y)\delta t + \ldots, \quad y_1 = y + \eta(x, y)\delta t + \ldots,$$

belongs to at least one one parameter group with inverse transformations, when infinitesimals of the second and higher orders are neglected. The finite equations of this group are found by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt,$$

with the initial conditions

$$x_1 = x, \quad y_1 = y, \quad \text{for } t=0,$$

in the form

$$\Omega(x_1, y_1) = \Omega(x, y),$$
$$W(x_1, y_1) - t = W(x, y);$$

or, solved with regard to x, y, and developed in powers of t, in the form

$$x_1 = x + \xi(x, y)\frac{t}{1!} + \left(\xi \frac{\partial \xi}{\partial x} + \eta \frac{\partial \xi}{\partial y} \right) \frac{t^2}{1!} + \ldots,$$
$$y_1 = y + \eta(x, y)\frac{t}{1!} + \left(\xi \frac{\partial \eta}{\partial x} + \eta \frac{\partial \eta}{\partial y} \right) \frac{t^2}{2!} + \ldots,$$

The one parameter group thus generated accordingly possesses an infinitesimal transformation which in its terms of the first order is identical with the original infinitesimal transformation.

We have now proved that every G_1 contains an infinitesimal transformation and conversely that every infinitesimal transformation generates a G_1. We shall prove in the next article that a G_1 contains but one infinitesimal transformation, with the converse that an infinitesimal transformation belongs to but one G_1. The theorems will be illustrated by concrete examples. These theorems establish the equivalence of the notions one parameter group and infinitesimal transformation; that these notions may be used interchangeably is the fundamental principle of LIE's Theory of the Group of One Parameter.

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[To be Continued.]